Explicit Conditions for the Convergence of Point Processes Associated to Stationary Arrays

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Abstract

In this article, we consider a stationary array $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ of random variables with values in $\mathbb{R}\backslash\{0\}$ (which satisfy some asymptotic dependence conditions), and the corresponding sequence $(N_n)_{n \geq 1}$ of point processes, where N_n has the points $X_{j,n}, 1 \leq j \leq n$. Our main result identifies some explicit conditions for the convergence of the sequence $(N_n)_{n \geq 1}$, in terms of the probabilistic behavior of the variables in the array.

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1 Introduction

The study of the asymptotic behavior of the sum (or the maximum) of the row variables in an array $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ is one of oldest problem in probability theory. When the variables are independent on each row, classical results identify the limit to have an infinitely divisible distribution in the case of the sum (see [9]), and a max-infinitely divisible distribution, in the case of the maximum (see [3]). A crucial observation, which can be traced back to [17], [22] (in the case of the maximum), and [20] (in the case of the sum) is that these results are deeply connected to the convergence in distribution of the sequence $N_n = \sum_{j=1}^{n} \delta_{X_{j,n}}, n \geq 1$ of point processes to a Poisson process N. (See Section 5.3 of [18] and Section 7.2 of [19], for a modern account on this subject.)

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Subsequent investigations showed that a similar connection exists in the case of arrays which possess a row-wise dependence structure (e.g. [8]). The most interesting case arises when $X_{i,n} = X_i/a_n$, where $(X_i)_{i\geq 1}$ is a (dependent) stationary sequence with regularly varying tails and $(a_n)_n$ is a sequence of real numbers such that $nP(|X_1| > a_n) \to 1$ (see [7] and the references therein). In the dependent case, the limit N may not be a Poisson process, but belongs to the class of infinitely divisible point processes (under generally weak assumptions). These findings reveal that the separate study of the point process convergence is an important topic, which may yield new asymptotic results for triangular arrays.

In the present article, we consider an array $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ whose row variables are asymptotically independent, in the sense that the block $(X_{1,n}, \ldots, X_{n,n})$ behaves asymptotically as k_n "smaller" i.i.d. blocks, a small block having the same distribution as $(X_{1,n}, \ldots, X_{r_n,n})$, with $n \sim r_n k_n$. This condition, that we call here (AD-1), was considered by many authors (e.g. [11] [12], [7], [10], [1]).

The rows of the array also possess an "anti-clustering" property (AC), which specifies the dependence structure within a small block. Intuitively, under (AC), it becomes improbable to find two points $X_{j,n}, X_{k,n}$ whose indices j, k are situated in the same small block at a distance larger than a fixed value m, and whose values (in modulus) exceed a fixed threshold $\varepsilon > 0$. Condition (AC) appeared, in various forms, in the literature related to the asymptotic behavior of the maximum (e.g. [14], [15]) or the sum (e.g. [5], [6], [4]). In addition, we assume the usual asymptotic negligibility (AN) condition for $X_{1,n}$.

Our main result says that under (AD-1), (AC) and (AN), the convergence $N_n \xrightarrow{d} N$, where N is an infinitely divisible point process, reduces to the convergence of:

$$nP(\max_{1\leq j\leq m-1} |X_{j,n}| \leq x, X_{m,n} > x), \text{ and} \tag{1}$$

$$n[P(A_{m,n}, \max_{1\leq j\leq m} |X_{j,n}| > x) - P(A_{m-1,n}, \max_{1\leq j\leq m-1} |X_{j,n}| > x)], \tag{2}$$

where $A_{m,n}$ is the event that at least k_i among $X_{1,n}, \ldots, X_{m,n}$ lie in B_i, for all $i = 1, \ldots, d$ (for arbitrary $d, k_1, \ldots, k_d \in \mathbb{N}$ and compact sets $B_1, \ldots, B_d$).

The novelty of this result compared to the existing results (e.g. Theorem 2.6 of [1]), is the fact that the quantities appearing in (1) and (2) speak *explicitly* about the probabilistic behavior of the variables in the array.

The article is organized as follows. In Section 2, we give the statements of the main result (Theorem 2.5) and a preliminary result (Theorem 2.4). Section 3 is dedicated to the proof of these two results. Section 4 contains a separate result about the extremal index of a stationary sequence, whose proof is related to some of the methods presented in this article.

2 The Main Results

We begin by introducing the terminology and the notation. Our main reference is [13]. We denote $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$.

If E is a locally compact Hausdorff space with a countable basis (LCCB), we let $\mathcal{B}$ be the class of all relatively compact Borel sets in E, and $C_K^+(E)$ be the class of continuous functions $f : E \to \mathbb{R}_+$ with compact support. We let $M_p(E)$ be the class of Radon measures on E with values in $\mathbb{Z}_+$ (endowed with the topology of vague convergence), and $\mathcal{M}_p(E)$ be the associated Borel σ-field. For $\mu \in M_p(E)$ and $f \in C_K^+(E)$, we denote $\mu(f) = \int_E f(x)\mu(dx)$. We denote by o the null measure.

Let $(\Omega, \mathcal{K}, P)$ be a probability space. A measurable map $N : \Omega \to M_p(E)$ is called a point process. Its distribution $P \circ N^{-1}$ is determined by the Laplace functional $L_N(f) = E(e^{-N(f)}), f \in C_K^+(E)$.

A point process N is *infinitely divisible* if for any $k \geq 1$, there exist some i.i.d. point processes $(N_{i,k})_{1 \leq i \leq k}$ such that $N \stackrel{d}{=} \sum_{i=1}^k N_{i,k}$. By Theorem 6.1 of [13], the Laplace functional of an infinitely divisible point process is given by:

$$L_N(f) = \exp\left\{-\int_{M_p(E)\setminus\{o\}} (1 - e^{-\mu(f)})\lambda(d\mu)\right\}, \quad \forall f \in C_K^+(E),$$

where λ is a measure on $M_p(E)\setminus\{o\}$, called the canonical measure of N.

All the point processes considered in this article have their points in $\mathbb{R}\setminus\{0\}$. For technical reasons, we embed $\mathbb{R}\setminus\{0\}$ into the space $E = [-\infty, \infty]\setminus\{0\}$. Let $\mathcal{B}$ be the class of relatively compact sets in E. Note that

$$[-x, x]^c := [-\infty, -x) \cup (x, \infty] \in \mathcal{B}, \quad \text{for all } x > 0.$$

We consider a triangular array $(X_{j,n})_{j \leq n, n \geq 1}$ of random variables with values in $\mathbb{R}\setminus\{0\}$, such that $(X_{j,n})_{j \leq n}$ is a strictly stationary sequence, for any $n \geq 1$.

Definition 2.1 The triangular array $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ satisfies:

(i) **condition (AN)** if

$$\limsup_{n\to\infty} nP(|X_{1,n}| > \varepsilon) < \infty, \quad \text{for all } \varepsilon > 0.$$

(ii) **condition (AD-1)** if there exists $(r_n)_n \subset \mathbb{N}$ with $r_n \to \infty$ and $k_n = [n/r_n] \to \infty$, such that:

$$\lim_{n\to\infty} \left| E\left(e^{-\sum_{j=1}^n f(X_{j,n})}\right) - \left\{E\left(e^{-\sum_{j=1}^{r_n} f(X_{j,n})}\right)\right\}^{k_n} \right| = 0, \quad \text{for all } f \in C_K^+(E).$$

(iii) **condition (AC)** if there exists $(r_n)_n \subset \mathbb{N}$ with $r_n \to \infty$, such that:

$$\lim_{m\to m_0} \limsup_{n\to\infty} n \sum_{j=m+1}^{r_n} P(|X_{1,n}| > \varepsilon, |X_{j,n}| > \varepsilon) = 0, \quad \text{for all } \varepsilon > 0,$$

where $m_0 := \inf\{m \in \mathbb{Z}_+; \lim_{n\to\infty} n \sum_{j=m+1}^{r_n} P(|X_{1,n}| > \varepsilon, |X_{j,n}| > \varepsilon) = 0, \text{ for all } \varepsilon > 0\}$. We use the conventions: $\inf \emptyset = \infty$ and $\lim_{m\to m_0} \phi(m) = \phi(m_0)$ if $m_0 < \infty$.

Remark 2.2 (i) For each $n \geq 1$, let $N_n = \sum_{j=1}^{n} \delta_{X_{j,n}}$ and $\tilde{N}_n = \sum_{i=1}^{k_n} \tilde{N}_{i,n}$, where $(\tilde{N}_{i,n})_{i \leq k_n}$ are i.i.d. copies of $N_{r_n,n} = \sum_{j=1}^{r_n} \delta_{X_{j,n}}$. Under (AD-1), $(N_n)_n$ converges in distribution if and only if $(\tilde{N}_n)_n$ does, and the limits are the same.

(ii) Condition (AN) is an asymptotic negligibility condition which ensures that $(\tilde{N}_{i,n})_{i \leq k_n, n \geq 1}$ is a *null-array* of point processes, i.e. $P(\tilde{N}_{1,n}(B) > 0) \to 0$ for all $B \in \mathcal{B}$. By Theorem 6.1 of [13] $\tilde{N}_n \xrightarrow{d} N$ if and only if

$$k_n E(1 - e^{-N_{r_n,n}(f)}) \to \int_{M_p(E)\setminus\{o\}} (1 - e^{-\mu(f)})\lambda(d\mu), \quad \forall f \in C_K^+(E).$$

In this case, N is an infinitely divisible point process with canonical measure λ.

Remark 2.3 (i) Condition (AD-1) is satisfied by arrays whose row-wise dependence structure is of mixing type (see e.g. Lemma 5.1 of [1]).

(ii) Condition (AC) is satisfied with $m_0 = m$ if $(X_{j,n})_{1 \leq j \leq n}$ is m-dependent.

(iii) When $X_{j,n} = X_j/u_n$ and $m_0 = 1$, condition (AC) is known in the literature as Leadbetter's condition $D'(\{u_n\})$ (see [14]).

(iii) A condition similar to (AC) was used in [4], [5] and [6] for obtaining the convergence of the partial sum sequence to an infinitely divisible random variable (with finite variance).

As in [7], let $M_0 = \{\mu \in M_p(\mathbb{R}\setminus\{0\}); \mu \neq o, \exists\, x \in (0,\infty) \text{ such that } \mu([-x,x]^c) = 0\}$. If $\mu = \sum_{j \geq 1} \delta_{t_j} \in M_0$, we let $x_\mu := \sup_{j \geq 1} |t_j| < \infty$. For each $x > 0$, let

$$M_x = \{\mu \in M_0; \mu([-x,x]^c) > 0\} = \{\mu \in M_0; x_\mu > x\}.$$

Recall that x is a *fixed atom* of a point process N if $P(N\{x\} > 0) > 0$. To simplify the writing, we introduce some additional notation. If $x > 0$ and λ is a measure on $M_p(E)$ with $\lambda(M_0^c) = 0$, we let

$$\mathcal{B}_{x,\lambda} = \{B \in \mathcal{B}; \lambda(\{\mu \in M_x; \mu(\partial B) > 0\}) = 0\},$$

and $\mathcal{J}_{x,\lambda}$ be the class of sets $M = \cap_{i=1}^{d}\{\mu \in M_p(E); \mu(B_i) \geq k_i\}$ for some $B_i \in \mathcal{B}_{x,\lambda}$, $k_i \geq 1$ (integers) and $d \geq 1$.

The following result is a refinement of Theorem 3.6 of [2].

Theorem 2.4 *Suppose that $(X_{j,n})_{1 \leq j \leq n, n \geq 1}$ satisfies (AN) and (AD-1) (with sequences $(r_n)_n$ and $(k_n)_n$). Let N be an infinitely divisible point process on $\mathbb{R}\setminus\{0\}$ with canonical measure λ. Let D be the set of fixed atoms of N and $D' = \{x > 0; x \in D \text{ or } -x \in D\}$.*

The following statements are equivalent:

(i) $N_n \xrightarrow{d} N$;

(ii) We have $\lambda(M_0^c) = 0$, and the following two conditions hold:

(a) $\quad k_n P(\max_{j \leq r_n} |X_{j,n}| > x) \to \lambda(M_x)$, *for any* $x > 0, x \notin D'$,

(b) $\quad k_n P(N_{r_n,n} \in M, \max_{j \leq r_n} |X_{j,n}| > x) \to \lambda(M \cap M_x)$, *for any* $x > 0, x \notin D'$ *and for any set* $M \in \mathcal{J}_{x,\lambda}$.

For each $1 \le m \le n$, let $N_{m,n} = \sum_{j=1}^{m} \delta_{X_{j,n}}$ and $M_{m,n} = \max_{j \le m} |X_{j,n}|$, with the convention that $M_{0,n} = 0$. The next theorem is the main result of this article, and gives an explicit form for conditions (a) and (b), under the additional anti-clustering condition (AC).

Theorem 2.5 *Let $(X_{j,n})_{1 \le j \le n, n \ge 1}$ and N be as in Theorem 2.4. Suppose in addition that (AC) holds, with the same sequence $(r_n)_n$ as in (AD-1).*

The following statements are equivalent:

(i) $N_n \xrightarrow{d} N$;

(ii) We have $\lambda(M_0^c) = 0$ and the following two conditions hold:

$$(a') \qquad \lim_{m \to m_0} \limsup_{n \to \infty} |n[P(M_{m,n} > x) - P(M_{m-1,n} > x)] - \lambda(M_x)| = 0, \ \textit{for any}\ x > 0, x \notin D',$$

$$(b') \qquad \lim_{m \to m_0} \limsup_{n \to \infty} |n[P(N_{m,n} \in M, M_{m,n} > x) - P(N_{m-1,n} \in M, M_{m-1,n} > x)] - \lambda(M \cap M_x)| = 0, \ \textit{for any}\ x > 0, x \notin D' \ \textit{and for any set}\ M \in \mathcal{J}_{x,\lambda}.$$

Remark 2.6 Note that

$$P(M_{m,n} > x) - P(M_{m-1,n} > x) = P(\max_{1 \le j \le m-1} |X_{j,n}| \le x, |X_{m,n}| > x).$$

Remark 2.7 Suppose that $m_0 = 1$ in Theorem 2.5. One can prove that in this case, the limit N is a Poisson process of intensity ν given by:

$$\nu(B) = \lambda(\{\mu \in M_p(E) \backslash \{o\}; \mu(B) = 1\}), \quad \forall B \in \mathcal{B}.$$

3 The Proofs

3.1 Proof of Theorem 2.4

Before giving the proof, we need some preliminary results.

Lemma 3.1 *Let E be a LCCB space and $M = \cap_{i=1}^{d} \{\mu \in M_p(E); \mu(B_i) \ge k_i\}$ for some $B_i \in \mathcal{B}$, $k_i \ge 1$ (integers) and $d \ge 1$. Then:*
(i) M is closed (with respect to the vague topology);
(ii) $\partial M \subset \cup_{i=1}^{d} \{\mu \in M_p(E); \mu(\partial B_i) > 0\}$.

Proof: Note that $\partial M \subset \cup_{i=1}^{d} \partial M_i$, where $M_i = \{\mu \in M_p(E); \mu(B_i) \ge k_i\}$. Since the finite intersection of closed sets is a closed set, it suffices to consider the case $d = 1$, i.e. $M = \{\mu \in M_p(E); \mu(B) \ge k\}$ for some $B \in \mathcal{B}$ and $k \ge 1$.

(i) Let $(\mu_n)_n \subset M$ be such that $\mu_n \xrightarrow{v} \mu$. If $\mu(\partial B) = 0$, then $\mu_n(B) \to \mu(B)$, and since $\mu_n(B) \ge k$ for all n, it follows that $\mu(B) \ge k$. If not, we proceed as in the proof of Lemma 3.15 of [18]. Let B^δ be a δ-swelling of B. Then $S = \{\delta \in (0, \delta_0]; \mu(\partial B^\delta) > 0\}$ is a countable set. By the previous argument, $\mu(B^\delta) \ge k$ for all $\delta \in (0, \delta_0] \backslash S$. Let $(\delta_n)_n \in (0, \delta_0] \backslash S$ be such that $\delta_n \downarrow 0$. Since $\mu(B^{\delta_n}) \ge k$ for all n, and $\mu(B^{\delta_n}) \downarrow \mu(B)$, it follows that $\mu(B) \ge k$, i.e. $\mu \in M$.

(ii) By part (i), $\partial M = \bar{M} \backslash M^o = M \cap (M^o)^c$. We will prove that $\partial M \subset \{\mu \in M; \mu(\partial B) > 0\}$, or equivalently

$$A := \{\mu \in M; \mu(\partial B) = 0\} \subset M^o.$$

Since M^o is the largest open set included in M and $A \subset M$, it suffices to show that A is open. Let $\mu \in A$ and $(\mu_n)_n \subset M_p(E)$ be such that $\mu_n \xrightarrow{v} \mu$. Then $\mu_n(B) \to \mu(B)$, and since $\mu(B) \geq k$ and $\{\mu_n(B)\}_n$ are integers, it follows that $\mu_n(B) \geq k$ for all $n \geq N_1$, for some N_1.

On the other hand, $\mu_n(\partial B) \to \mu(\partial B)$, since $\partial B \in \mathcal{B}$ and $\mu(\partial B) = 0$ (note that $\partial(\partial B) = \partial B$). Since $\mu(\partial B) = 0$ and $\{\mu_n(\partial B)\}_n$ are integers, it follows that $\mu_n(\partial B) = 0$ for all $n \geq n_2$, for some n_2. Hence $\mu_n \in A$ for all $n \geq \max\{n_1, n_2\}$. $\square$

Lemma 3.2 *Let E be a LCCB space and $(Q_n)_n, Q$ be probability measures on $M_p(E)$. Let $\mathcal{B}_Q$ be the class of all sets $B \in \mathcal{B}$ which satisfy:*

$$Q(\{\mu \in M_p(E); \mu(\partial B) > 0\}) = 0,$$

and $\mathcal{J}_Q$ be the class of sets $M = \cap_{i=1}^d \{\mu \in M_p(E); \mu(B_i) \geq k_i\}$ for some $B_i \in \mathcal{B}_Q$, $k_i \geq 1$ (integers) and $d \geq 1$.

Then $Q_n \xrightarrow{w} Q$ if and only if $Q_n(M) \to Q(M)$ for all $M \in \mathcal{J}_Q$.

Proof: Let $(N_n)_n, N$ be point processes on E, defined on a probability space $(\Omega, \mathcal{F}, P)$, such that $P \circ N_n^{-1} = Q_n$ for all n, and $P \circ N^{-1} = Q$. Note that $\mathcal{B}_Q = \mathcal{B}_N := \{B \in \mathcal{B}; N(\partial B) = 0 \text{ a.s.}\}$.

By definition, $N_n \xrightarrow{d} N$ if and only if $Q_n \xrightarrow{w} Q$. By Theorem 4.2 of [13], $N_n \xrightarrow{d} N$ if and only if

$$(N_n(B_1), \ldots, N_n(B_d)) \xrightarrow{d} (N(B_1), \ldots, N(B_d))$$

for any $B_1, \ldots, B_d \in \mathcal{B}_N$ and for any $d \geq 1$. Since these random vectors have values in $\mathbb{Z}_+^d$, the previous convergence in distribution is equivalent to:

$$P(N_n(B_1) = k_1, \ldots, N_n(B_d) = k_d) \to P(N(B_1) = k_1, \ldots, N(B_d) = k_d)$$

for any $k_1, \ldots, k_d \in \mathbb{Z}_+$, which is in turn equivalent to

$$P(N_n(B_1) \geq k_1, \ldots, N_n(B_d) \geq k_d) \to P(N(B_1) \geq k_1, \ldots, N(B_d) \geq k_d)$$

for any $k_1, \ldots, k_d \in \mathbb{Z}_+$. Finally, it suffices to consider only integers $k_i \geq 1$ since, if there exists a set $I \subset \{1, \ldots, d\}$ such that $k_i = 0$ for all $i \in I$ and $k_i \geq 1$ for $i \notin I$, then $P(N_n(B_1) \geq k_1, \ldots, N_n(B_d) \geq k_d) = P(N_n(B_i) \geq k_i, i \notin I) \to P(N(B_i) \geq k_i, i \notin I) = P(N(B_1) \geq k_1, \ldots, N(B_d) \geq k_d)$. $\square$

Proof of Theorem 2.4: Note that $\{\max_{j \leq r_n} |X_{j,n}| > x\} = \{N_{r_n,n} \in M_x\}$.

Suppose that (i) holds. As in the proof of Theorem 3.6 of [2], it follows that $\lambda(M_0^c) = 0$ and (a) holds. Moreover, we have $P_{n,x} \stackrel{w}{\to} P_x$ where $P_{n,x}$ and P_x are probability measures on $M_p(E)$ defined by:

$$P_{n,x}(M) = \frac{k_n P(N_{r_n,n} \in M \cap M_x)}{k_n P(N_{r_n,n} \in M_x)} \quad \text{and} \quad P_x(M) = \frac{\lambda(M \cap M_x)}{\lambda(M_x)}.$$

Therefore, $P_{n,x}(M) \to P_x(M)$ for any $M \in \mathcal{M}_p(E)$ with $P_x(\partial M) = 0$. Since $k_n P(N_{r_n,n} \in M_x) \to \lambda(M_x)$ (by (a)), it follows that

$$k_n P(N_{r_n,n} \in M \cap M_x) \to \lambda(M \cap M_x), \tag{3}$$

for any $M \in \mathcal{M}_p(E)$ with $\lambda(\partial M \cap M_x) = 0$.

In particular, (3) holds for a set $M = \cap_{i=1}^d \{\mu \in M_p(E); \mu(B_i) \geq 1\}$, with $B_i \in \mathcal{B}_{x,\lambda}$, $k_i \geq 1$ (integers) and $d \geq 1$. To see this, note that by Lemma 3.1, $\partial M \cap M_x \subset \cup_{i=1}^d \{\mu \in M_x; \mu(\partial B_i) > 0\}$, and hence

$$\lambda(\partial M \cap M_x) \leq \sum_{i=1}^d \lambda(\{\mu \in M_x; \mu(\partial B_i) > 0\}) = 0.$$

Suppose that (ii) holds. As in the proof of Theorem 3.6 of [2], it suffices to show that $P_{n,x} \stackrel{w}{\to} P_x$. This follows by Lemma 3.2, since the class of sets $B \in \mathcal{B}$ which satisfy:

$$P_x(\{\mu \in M_p(E); \mu(\partial B) > 0\}) = 0$$

coincides with $\mathcal{B}_{x,\lambda}$. □

3.2 Proof of Theorem 2.5

We begin with an auxiliary result, which is of independent interest.

Lemma 3.3 *Let $h : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function, such that*

$$\|D^2 h\|_\infty := \max_{i,j=1,\ldots,d} \sup_{\mathbf{x} \in \mathbb{R}^d} \left| \frac{\partial^2 h}{\partial x_i \partial x_j}(\mathbf{x}) \right| < \infty. \tag{4}$$

Let $(\mathbf{Y}_i)_{i \geq 1}$ be a strictly stationary sequence of d-dimensional random vectors with $\mathbf{Y}_i = (Y_i^{(1)}, \ldots, Y_i^{(d)})$. Let $\mathbf{S}_n = \sum_{i=1}^n \mathbf{Y}_i$ for $n \geq 1$ and $\mathbf{S}_0 = \mathbf{0}$. Then for any $1 \leq m \leq r$,

$$|E[h(\mathbf{S}_r)] - rE[h(\mathbf{S}_m) - h(\mathbf{S}_{m-1})]| \leq m|E[h(\mathbf{S}_m)] + E[h(\mathbf{S}_{m-1})]| +$$

$$\|D^2 h\|_\infty \sum_{k=0}^{r_n - m} \sum_{i,l=1}^d E|S_k^{(i)} Y_{k+m}^{(l)}|.$$

Proof: As in Lemma 3.2 of [12] (see also Theorem 2.6 of [1]), we have:

$$\begin{aligned} E[h(\mathbf{S}_r)] &= E[h(\mathbf{S}_{m-1})] + \sum_{k=0}^{r-m} E[h(\mathbf{S}_{k+m}) - h(\mathbf{S}_{k+m-1})] \\ rE[h(\mathbf{S}_m) - h(\mathbf{S}_{m-1})] &= (m-1)E[h(\mathbf{S}_m) - h(\mathbf{S}_{m-1})] + \\ & \sum_{k=0}^{r-m} E[h(\mathbf{S}_{k+m} - \mathbf{S}_k) - h(\mathbf{S}_{k+m-1} - \mathbf{S}_k)], \end{aligned}$$

where the second equality is due to the stationarity of $(\mathbf{Y}_i)_i$. Taking the difference, we get:

$$E[h(\mathbf{S}_r)] - rE[h(\mathbf{S}_m) - h(\mathbf{S}_{m-1})] = mE[h(\mathbf{S}_{m-1})] - (m-1)E[h(\mathbf{S}_m)] +$$
$$\sum_{k=0}^{r-m} E\{[h(\mathbf{S}_{k+m}) - h(\mathbf{S}_{k+m} - \mathbf{S}_k)] - [h(\mathbf{S}_{k+m-1}) - h(\mathbf{S}_{k+m-1} - \mathbf{S}_k)]\} =: I_1 + I_2.$$

Clearly $|I_1| \leq m|E[h(\mathbf{S}_{m-1})] + E[h(\mathbf{S}_m)]|$. For treating I_2, we use the Taylor's formula (with integral remainder) for twice continuously differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$:

$$f(x) - f(x_0) = \sum_{i=1}^{d} (x^{(i)} - x_0^{(i)}) \int_0^1 \frac{\partial f}{\partial x_i}(x - s(x - x_0))ds. \tag{5}$$

We get:

$$\begin{aligned} h(\mathbf{S}_{k+m}) - h(\mathbf{S}_{k+m} - \mathbf{S}_k) &= \sum_{i=1}^{d} S_k^{(i)} \int_0^1 \frac{\partial h}{\partial x_i}(\mathbf{S}_{k+m} - x\mathbf{S}_k)dx, \\ h(\mathbf{S}_{k+m-1}) - h(\mathbf{S}_{k+m-1} - \mathbf{S}_k) &= \sum_{i=1}^{d} S_k^{(i)} \int_0^1 \frac{\partial h}{\partial x_i}(\mathbf{S}_{k+m-1} - x\mathbf{S}_k)dx. \end{aligned}$$

Taking the difference of the last two equations, and using (5) for $f = \partial h / \partial x_i$ with $i = 1, \ldots, d$, we obtain:

$$\begin{aligned} & [h(\mathbf{S}_{k+m}) - h(\mathbf{S}_{k+m} - \mathbf{S}_k)] - [h(\mathbf{S}_{k+m-1}) - h(\mathbf{S}_{k+m-1} - \mathbf{S}_k)] = \\ & \quad \sum_{i=1}^{d} S_k^{(i)} \int_0^1 \left[\frac{\partial h}{\partial x_i}(\mathbf{S}_{k+m} - x\mathbf{S}_k) - \frac{\partial h}{\partial x_i}(\mathbf{S}_{k+m-1} - x\mathbf{S}_k) \right] dx \\ & = \sum_{i=1}^{d} S_k^{(i)} \int_0^1 \sum_{l=1}^{d} Y_{k+m}^{(l)} \int_0^1 \frac{\partial^2 h}{\partial x_i \partial x_l}((\mathbf{S}_{k+m} - x\mathbf{S}_k) - \theta \mathbf{Y}_{k+m}) d\theta dx. \end{aligned}$$

From here we conclude that:

$$|[h(\mathbf{S}_{k+m}) - h(\mathbf{S}_{k+m} - \mathbf{S}_k)] - [h(\mathbf{S}_{k+m-1}) - h(\mathbf{S}_{k+m-1} - \mathbf{S}_k)]| \leq \|D^2 h\|_\infty \sum_{i,l=1}^{d} |S_k^{(i)} Y_{k+m}^{(l)}|,$$

which yields the desired estimate for I_2. $\square$

Proposition 3.4 *Let E be a LCCB space. For each $n \geq 1$, let $(X_{j,n})_{j\leq n}$ be a strictly stationary sequence of E-valued random variables, such that:*

$$\limsup_{n\to\infty} nP(X_{1,n} \in B) < \infty, \text{ for all } B \in \mathcal{B}. \tag{6}$$

Suppose that there exists $(r_n)_n \subset \mathbb{N}$ with $r_n \to \infty$ and $k_n = [n/r_n] \to \infty$, such that:

$$\lim_{m\to m_0} \limsup_{n\to\infty} n \sum_{j=m+1}^{r_n} P(X_{1,n} \in B, X_{j,n} \in B) = 0, \text{ for all } B \in \mathcal{B}, \tag{7}$$

where $m_0 =: \{m \in \mathbb{Z}_+; \lim_{n\to\infty} n \sum_{j=m+1}^{r_n} P(X_{1,n} \in B, X_{j,n} \in B) = 0, \text{ for all } B \in \mathcal{B}\}$. Let $N_{m,n} = \sum_{j=1}^{m} \delta_{X_{j,n}}$. Then

$$\lim_{m\to m_0} \limsup_{n\to\infty} |k_n P(N_{r_n,n} \in M) - n[P(N_{m,n} \in M) - P(N_{m-1,n} \in M)]| = 0,$$

for any set $M = \cap_{i=1}^{d} \{\mu \in M_p(E); \mu(B_i) \geq k_i\}$, with $B_i \in \mathcal{B}$, $k_i \geq 1$ (integers) and $d \geq 1$.

Proof: Let $h : \mathbb{R}_+^d \to \mathbb{R}_+$ be a twice continuously differentiable function which satisfies (4), such that $h(x_1, \ldots, x_d) \leq x_1 + \ldots + x_d$ for all $(x_1, \ldots, x_d) \in \mathbb{R}_+^d$, and

$$h(x_1, \ldots, x_d) = \begin{cases} 0 & \text{if } x_i \leq k_i - 1 \text{ for some } i = 1, \ldots, d \\ 1 & \text{if } x_i \geq k_i \text{ for all } i = 1, \ldots, d \end{cases}$$

Note that:

$$h(x_1, \ldots, x_d) = 1_{\{x_1 \geq k_1, \ldots, x_d \geq k_d\}} \quad \text{for all } x_1, \ldots, x_d \in \mathbb{Z}_+. \tag{8}$$

For any $n \geq 1$, we consider strictly stationary sequence of d-dimensional random vectors $\{\mathbf{Y}_{j,n} = (Y_{j,n}^{(1)}, \ldots, Y_{j,n}^{(d)}), 1 \leq j \leq n\}$ defined by:

$$Y_{j,n}^{(i)} = 1_{\{X_{j,n} \in B_i\}}, \quad \text{for any } i = 1, \ldots, d.$$

Using (8), we obtain for any $1 \leq m \leq n$,

$$\begin{aligned} &P(N_{m,n} \in M) = P(N_{m,n}(B_1) \geq k_1, \ldots, N_{m,n}(B_d) \geq k_d) = \\ &\quad P(\sum_{j=1}^{m} Y_{j,n}^{(1)} \geq k_1, \ldots, \sum_{j=1}^{m} Y_{j,n}^{(d)} \geq k_d) = E[1_{\{\sum_{j=1}^{m} Y_{j,n}^{(1)} \geq k_1, \ldots, \sum_{j=1}^{m} Y_{j,n}^{(d)} \geq k_d\}}] = \\ &\quad E[h(\sum_{j=1}^{m} Y_{j,n}^{(1)}, \ldots, \sum_{j=1}^{m} Y_{j,n}^{(d)})] = E[h(\sum_{j=1}^{m} \mathbf{Y}_{j,n})]. \end{aligned}$$

Using Lemma 3.3, and letting $C = \|D^2 h\|_\infty$, we obtain:

$$\begin{aligned} &k_n |P(N_{r_n,n} \in M) - r_n[P(N_{m,n} \in M) - P(N_{m-1,n} \in M)]| \leq \\ &\quad m k_n \{E[h(\sum_{j=1}^{m} \mathbf{Y}_{j,n})] + E[h(\sum_{j=1}^{m-1} \mathbf{Y}_{j,n})]\} + C k_n \sum_{i,l=1}^{d} \sum_{k=0}^{r_n - m} E(\sum_{j=1}^{k} Y_{j,n}^{(i)} Y_{k+m,n}^{(l)}) \\ &\quad =: I_{m,n}^{(1)} + C I_{m,n}^{(2)} \end{aligned} \tag{9}$$

Using the fact that $h(\mathbf{x}) \leq \sum_{i=1}^d x_i$, and the stationary of $(X_{j,n})_{j\leq n}$,

$$\begin{aligned} I_{m,n}^{(1)} &\leq 2mk_n E(\sum_{j=1}^m \sum_{i=1}^d Y_{j,n}^{(i)}) = 2mk_n \sum_{i=1}^d \sum_{j=1}^m P(X_{j,n} \in B_i) \\ &= 2m^2 k_n \sum_{i=1}^d P(X_{1,n} \in B_i) \leq 2m^2 \frac{1}{r_n} \sum_{i=1}^d nP(X_{1,n} \in B_i). \end{aligned}$$

From (6), it follows that $\lim_{n\to\infty} I_{m,n}^{(1)} = 0$ for all m, and hence

$$\lim_{m\to m_0} \limsup_{n\to\infty} I_{m,n}^{(1)} = 0. \tag{10}$$

Using the stationarity of $(X_{j,n})_{j\leq n}$, and letting $B = \cup_{i=1}^d B_i \in \mathcal{B}$,

$$\begin{aligned} I_{m,n}^{(2)} &= k_n \sum_{i,l=1}^d \sum_{k=0}^{r_n-m} \sum_{j=1}^k P(X_{j,n} \in B_i, X_{k+m,n} \in B_l) \\ &= k_n \sum_{i,l=1}^d \sum_{j=m+1}^{r_n} (r_n - j + 1) P(X_{1,n} \in B_i, X_{j,n} \in B_l) \\ &\leq n \sum_{i,l=1}^d \sum_{j=m+1}^{r_n} P(X_{1,n} \in B_i, X_{j,n} \in B_l) \\ &\leq d^2 n \sum_{j=m+1}^{r_n} P(X_{1,n} \in B, X_{j,n} \in B). \end{aligned}$$

From (7), it follows that:

$$\lim_{m\to m_0} \limsup_{n\to\infty} I_{m,n}^{(2)} = 0. \tag{11}$$

From, (9), (10) and (11), it follows that:

$$\lim_{m\to m_0} \limsup_{n\to\infty} k_n |P(N_{r_n,n} \in M) - r_n[P(N_{m,n} \in M) - P(N_{m-1,n} \in M)]| = 0.$$

Note that $\lim_{n\to\infty}(n - k_n r_n)|P(N_{m,n} \in M) - P(N_{m-1,n} \in M)| = 0$ for all m, and hence

$$\lim_{m\to m_0} \limsup_{n\to\infty} (n - k_n r_n)|P(N_{m,n} \in M) - P(N_{m-1,n} \in M)| = 0.$$

The conclusion follows. $\Box$

Corollary 3.5 *For each $n \geq 1$, let $(X_{j,n})_{1\leq j\leq n}$ be a strictly stationary sequence of random variables with values in $\mathbb{R}\backslash\{0\}$. Suppose that $(X_{j,n})_{1\leq j\leq n, n\geq 1}$ satisfies (AN) and (AC).*

For any $1 \leq m \leq n$, let $N_{m,n} = \sum_{j=1}^{m} \delta_{X_{j,n}}$ and $M_{m,n} = \max_{j \leq m} |X_{j,n}|$. Then,

$$\lim_{m \to m_0} \limsup_{n \to \infty} |k_n P(M_{r_n,n} > x) - n[P(M_{m,n} > x) - P(M_{m-1,n} > x)]| = 0$$

$$\lim_{m \to m_0} \limsup_{n \to \infty} |k_n P(N_{r_n,n} \in M, M_{r_n,n} > x) - n[P(N_{m,n} \in M, M_{m,n} > x) - P(N_{m-1,n} \in M, M_{m-1,n} > x)]| = 0,$$

for any $x > 0$, and for any set $M = \cap_{i=1}^{d} \{\mu \in M_p(E); \mu(B_i) \geq k_i\}$, with $B_i \in \mathcal{B}$, $k_i \geq 1$ (integers) and $d \geq 1$.

Proof: Since $\{M_{m,n} > x\} = \{N_{m,n}([-x,x]^c) \geq 1\}$ for any $1 \leq m \leq n$, the result follows from Proposition 3.4. □

Proof of Theorem 2.5: The result follows from Theorem 2.4 and Corollary 3.5. □

4 The extremal index

In this section, we give a recipe for calculating the extremal index of a stationary sequence, using a method which is similar to that used for proving Theorem 2.5, in a simplified context. Although this recipe (given by Theorem 4.5 below) seems to be known in the literature (see [15], [16], [21]), we decided to include it here, since we could not find a direct reference for its proof.

We recall the following definition.

Definition 4.1 Let $(X_j)_{j \geq 1}$ be a strictly stationary sequence of random variables. **The extremal index** of the sequence $(X_j)_{j \geq 1}$, if it exists, is a real number θ with the following property: for any $\tau > 0$, there exists a sequence $(u_n^{(\tau)})_n \subset \mathbb{R}$ such that $nP(X_1 > u_n^{(\tau)}) \to \tau$ and $P(\max_{j \leq n} X_j \leq u_n^{(\tau)}) \to e^{-\tau\theta}$.

In particular, for $\tau = 1$, we denote $u_n^{(1)} = u_n$, and we have

$$nP(X_1 > u_n) \to 1 \quad \text{and} \quad P(\max_{j \leq n} X_j \leq u_n) \to e^{-\theta}. \tag{12}$$

It is clear that if it exists, $\theta \in [0, 1]$.

Remark 4.2 The extremal index of an i.i.d sequence exists and is equal to 1.

The following definition was originally considered in [15].

Definition 4.3 We say that $(X_j)_{j \geq 1}$ satisfies **condition (AIM)** (or admits an asymptotic independence representation for the maximum) if there exists $(r_n)_n \subset \mathbb{N}$ with $r_n \to \infty$ and $k_n = [n/r_n] \to \infty$, such that:

$$\left| P(\max_{j \leq n} X_j \leq u_n) - P(\max_{j \leq r_n} X_j \leq u_n)^{k_n} \right| \to 0.$$

Remark 4.4 It is known that (Leadbetter's) condition $D(\{u_n\})$ implies (AIM) (see Lemma 2.1 of [14]). Recall that $(\xi_j)_j$ satisfies condition $D(\{u_n\})$ if there exists a sequence $(m_n)_n \subset \mathbb{N}$, such that $m_n = o(n)$ and $\alpha_n(m_n) \to 0$, where

$$\alpha_n(m) = \sup_{I,J} |P(\max_{j\in I} X_j \le u_n, \max_{j\in J} X_j \le u_n) - P(\max_{j\in I} X_j \le u_n)P(\max_{j\in J} X_j \le u_n)|,$$

where the supremum ranges over all disjoint subsets I, J of $\{1, \ldots, n\}$, which are separated by a block of length greater of equal than m.

The following theorem is the main result of this section.

Theorem 4.5 *Let $(X_j)_{j\ge1}$ be a strictly stationary sequence whose extremal index θ exists, and $(u_n)_n$ be a sequence of real numbers satisfying (12).*

Suppose that $(X_j)_{j\ge1}$ satisfies (AIM), and in addition,

$$\lim_{m\to m_0} \limsup_{n\to\infty} n \sum_{j=m+1}^{r_n} P(X_1 > u_n, X_j > u_n) = 0, \tag{13}$$

where $m_0 := \inf\{m \in \mathbb{Z}_+; \lim_{n\to\infty} n \sum_{j=m+1}^{r_n} P(X_1 > u_n, X_j > u_n) = 0\}$.

Then

$$\theta = \lim_{m\to m_0} \limsup_{n\to\infty} nP(\max_{1\le j\le m-1} X_j \le u_n, X_m > u_n). \tag{14}$$

Due to the stationarity, and the fact that $nP(X_1 > u_n) \to 1$, (14) can be written as:

$$\begin{aligned} \theta &= \lim_{m\to m_0} \limsup_{n\to\infty} nP(\max_{2\le j\le m} X_j \le u_n, X_1 > u_n) \\ &= \lim_{m\to m_0} \limsup_{n\to\infty} P(\max_{2\le j\le m} X_j \le u_n | X_1 > u_n), \end{aligned}$$

which coincides with (2.3) of [21].

Remark 4.6 Let $(Y_i)_{i\ge1}$ be a sequence of i.i.d. random variables and $X_i = \max(Y_i, \cdots, Y_{i+m-1})$. Then $(X_i)_{i\ge1}$ satisfies condition (13), since it is an m-dependent sequence. A direct calculation shows that the extremal index of $(X_i)_{i\ge1}$ exists and is equal to $1/m$, which can be deduced also from (14).

The proof of Theorem 4.5 is based on some intermediate results.

Proposition 4.7 *Let $(X_j)_{j\ge1}$ be a strictly stationary sequence whose extremal index θ exists, and $(u_n)_n$ be a sequence of real numbers satisfying (12). If $(X_j)_{j\ge1}$ satisfies (AIM), then $k_n P(\max_{j\le r_n} X_j > u_n) \to \theta$.*

Proof: Due to (AIM), $P(\max_{j\le r_n} X_j \le u_n)^{k_n} \to e^{-\theta}$. The result follows, since

$$P(\max_{j\le r_n} X_j \le u_n)^{k_n} = \left(1 - \frac{k_n P(\max_{j\le r_n} X_j > u_n)}{k_n}\right)^{k_n}.$$

□

Proposition 4.8 *Let $(X_j)_{j\geq 1}$ be a strictly stationary sequence such that:*

$$\limsup_{n\to\infty} nP(X_1 > u_n) < \infty.$$

Suppose that there exists $(r_n)_n \subset \mathbb{N}$ with $r_n \to \infty$ and $k_n = [n/r_n] \to \infty$, such that (13) holds. Then

$$\lim_{m\to m_0} \limsup_{n\to\infty} |k_n P(\max_{j\leq r_n} X_j > u_n) - n[P(\max_{j\leq m} X_j > u_n) - P(\max_{j\leq m-1} X_j > u_n)]| = 0.$$

Proof: The argument is the same as in the proof of Proposition 3.4, using Lemma 3.3. More precisely, we let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a twice continuously differentiable such that $\|h''\|_\infty < \infty$, $h(0) = 0$, $h(1) = 1$ if $y \geq 1$, and $h(x) \leq x$ for all $x \in \mathbb{R}_+$. Then $h(x) = 1_{\{x\geq 1\}}$ for all $x \in \mathbb{Z}_+$, and

$$P(\max_{j\leq m} X_j > u_n) \quad = \quad E[1_{\{\sum_{j=1}^m 1_{\{X_j > u_n\}} \geq 1\}}] = E[h(\sum_{j=1}^m 1_{\{X_j > u_n\}})].$$

We omit the details. □

Proof of Theorem 4.5: The result follows from Proposition 4.7 and Proposition 4.8, using the fact that:

$$P(\max_{j\leq m-1} X_j \leq u_n, X_m > u_n) = P(\max_{j\leq m} X_j > u_n) - P(\max_{j\leq m-1} X_j > u_n).$$

□

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